J Glob Optim (2008) 41:43–60
DOI 10.1007/s10898-007-9165-x

ORIGINAL PAPER

Characterizing efficiency without linear structure: a unified approach

Fabián Flores-Bazán · Elvira Hernández · Vicente Novo

Received: 1 November 2006 / Accepted: 10 April 2007 / Published online: 22 May 2007

Abstract In this paper, we study a general optimization problem without linear structure under a reflexive and transitive relation on a nonempty set E, and characterize the existence of efficient points and the domination property for a subset of E through a generalization of the order-completeness condition introduced earlier. Afterwards, we study the abstract optimization problem by using generalized continuity concepts and establish various existence results. As an application, we extend and improve several existence results given in the literature for an optimization problem involving set-valued maps under vector and set criteria.

Keywords Efficiency · Vector optimization · Set optimization · Existence results · Domination property

Mathematics Subject Classification (2000) 90C29 · 90C48 · 90C30 · 54C60

The work of the first author was partially supported by CONICYT-Chile through FONDECYT 107-0689 and FONDAP-Matemáticas Aplicadas II; whereas that of the second and third author were supported in part by Ministerio de Educación y Ciencia (Spain), project MTM2006-02629 and by Junta de Castilla y León (Spain) Project VA027B06.

F. Flores-Bazán (✉)
Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
e-mail: fflores@ing-mat.udec.cl

E. Hernández · V. Novo
Departamento de Matemática Aplicada, Universidad Nacional de Educación a Distancia, c/ Juan del Rosal, 12, 28040 Madrid, Spain

E. Hernández
e-mail: ehernandez@ind.uned.es

V. Novo
e-mail: vnovo@ind.uned.es

1 Introduction

Existence results of efficient points of a set are of great interest in optimization theory. Many authors have obtained sufficient conditions for the existence of efficient points in the framework of topological vector spaces ordered by a convex cone, such as Yu [23], Wagner [21], Hartley [9], Corley [3], Luc [16], [17], Ferro [5], [6], Jahn [12], Zhu et al. [22] and references therein. Hartley established cone-compactness as a sufficient condition for the existence of efficient points. Corley studied cone-semicompactness and used it to give existence results. Luc introduced the notion of cone-completeness in order to characterize the existence of efficient points of a set and to extend several existence results of vector optimization. Ferro in his nice work [5] besides revising some of the results in [4] and [16], provided various existence theorems improving and generalizing previous results when the partial order comes from a convex, closed and pointed cone.

One may ask whether these notions (cone-semicompactness and cone-completeness) and existence results are still true if the linear structure does not exist. Furthermore, is it possible to consider these concepts in a set equipped with a reflexive and transitive relation?.

About this question, in [19] the authors discussed the existence of efficient points of a set with respect to general transitive relations and in [18], without linear structure, an existence result of efficient points is given.

The aim of this paper is to show that some known existence results of efficient points in vector optimization do not rely on the linearity of the space.

An important application of the topic of this paper regards certain set-valued optimization problems as we now briefly describe.

Let X be a nonempty set and Y be a vector space equipped with a partial order $\leq$. Associated to a given set-valued map $F: X \to \wp_0(Y)$, where $\wp_0(Y) \doteq 2^Y \setminus \{\emptyset\}$, the vector optimization problem with set-valued maps is usually defined as follows

$$(VP) \qquad \text{Min } F(x) \text{ subject to } x \in X,$$

which means to find $x_0 \in X$ with the property that there exists $y_0 \in F(x_0)$ such that

$$y_0 \in \text{Min}(F(X), \leq)$$

where $F(X) = \bigcup_{x \in X} F(x)$. This notion says, roughly speaking, that x_0 is a solution to (VP) if $F(x_0)$ contains an efficient element of $F(X)$ with respect to the ordering $\leq$ defined on Y. This type of criterion of solution is called *vector criterion*. However, sometimes it is more desirable the existence of some x_0 such that the whole set $F(x_0)$ is an efficient element of the family of subsets $\{F(x): x \in X\}$ with respect to a preference $\preccurlyeq$ given by a reflexive and transitive relation defined on $\wp_0(Y)$. This criterion of solution called *set criterion* was introduced by Kuroiwa in 1998. So, in this framework, the efficient notions are defined by means a partial order involving sets instead of points. Therefore, there is no linear structure due to the partial order $\preccurlyeq$ is defined on $\wp_0(Y)$. Several concepts taking into account the previous observation have been first introduced by Kuroiwa in [13]. Recently, more and more authors investigate set-valued optimization problems where the solutions are obtained by means of set relations (see, for example, [8], [14], [15], [1], [7], [10] and [11]). Moreover, we emphasize that set relations are widely used in theoretical computer science as well as in fixed point theory (see [2]).

The main goal of the present paper is to provide a general framework which encompasses the one employed in all the preceding papers. In particular, an abstract existence theorem under minimal assumptions is established. To that end, we consider any nonempty set E with a partial order, denoted by $\preccurlyeq$, which is reflexive and transitive.

An interesting example of nonvector optimization is answer set programming which has proven to be extremely useful for solving several artificial intelligence problems, see for example [20] and the references given there.

The outline of the present paper is as follows. Section 2 states several basic definitions and establishes various of their consequences which will be applied in the subsequent sections. In Sect. 3, we provide first, several characterizations for the existence of efficient points of any given set without linear structure under a reflexive and transitive relation, as well as some characterizations when the set is equipped with a topology. As applications, when the partial order comes from a convex cone K (so a linear structure is needed) we extend Theorem 3.4, Chapter 2 in [16] which provides a characterization under strongly K-completeness. On the other hand, when the partial order is not necessarily induced by a convex cone, we prove that the condition of Theorem 2.1 in [18] in terms of order s-semicompactness is also necessary. The domination property is characterized in Sect. 4 in terms of the notions introduced in Sect. 2. In Sect. 5 we establish conditions for the existence of solutions of a general optimization problem under continuity assumptions. Finally, as an application of the previous sections we study solutions of a set-valued optimization problem by using both criteria of solutions and improve some of the results obtained in [1], [14], [15] and [11]. The set criterion case is discussed in Sect. 6 and the vector criterion case in Sect. 7.

2 Preliminaries and basic definitions

In the sequel E denotes a nonempty set with a partial order $\preccurlyeq$ and A denotes a nonempty subset of E. We denote by A^c the complementary set of A. If E is a topological space we denote by $\operatorname{cl} A$ the closure of A.

A point $\bar{a} \in A$ is called an efficient point of A if

$$a \in A\,, a \preccurlyeq \bar{a} \Longrightarrow \bar{a} \preccurlyeq a.$$

The set of all efficient points of A is denoted by $\operatorname{Min}(A, \preccurlyeq)$. If $x,\ y \in E$ we denote by $x \sim y$ if and only if $x \preccurlyeq y$ and $y \preccurlyeq x$. We follow the notations used in [16]. For $x \in E$, the lower and upper section at x are defined by

$$L_x \doteq \{y \in E :\ y \preccurlyeq x\},\ S_x \doteq \{y \in E :\ x \preccurlyeq y\},$$

and

$$S_A \doteq \bigcup_{x \in A} S_x.$$

Thus, we can write

$$\operatorname{Min}(A, \preccurlyeq) = \{\bar{a} \in A :\ A \cap L_{\bar{a}} \subseteq S_{\bar{a}}\}.$$

On the other hand, it is not difficult to check that $S_{L_x^c} = L_x^c$ (see [18]), where $L_x^c \doteq (L_x)^c$. In case E is a vector space and the partial order is induced by a convex cone K, denoted by $\preccurlyeq_K$, simple computations show that

$$L_x = x - K,\ S_x = x + K,\ S_A = A + K.$$

For $A, B \subseteq E$, we say that A *is minorized by* B if, for each $a \in A$ there exists $b \in B$ such that $b \preccurlyeq a$, that is, if $A \subseteq S_B$. By a *lower section* of A we mean a set of the form

$$A_x \doteq L_x \cap A,\ \text{for some } x \in E.$$

The following proposition generalizes Lemmas 4.7 and 6.2(a) of [12] which are formulated in terms of vector optimization.

Proposition 2.1

(a) $\mathrm{Min}(A, \preccurlyeq) \subseteq \mathrm{Min}(S_A, \preccurlyeq)$. *Furthermore, if* $L_a \cap S_a = \{a\}$ *for all* $a \in A$ *then* $\mathrm{Min}(S_A, \preccurlyeq) = \mathrm{Min}(A, \preccurlyeq)$.
(b) *Let* $x \in E$. *Then* $\mathrm{Min}(A_x, \preccurlyeq) \subseteq \mathrm{Min}(A, \preccurlyeq)$.

We recall that a subset A of E is a totally ordered set or a chain in E if for all $x, y \in A$ either $x \preccurlyeq y$ or $y \preccurlyeq x$ is true.

The following two concepts are new and will play a fundamental role in our existence theory to be developed in the subsequent sections.

Definition 2.2

(a) *We say that* A *is order-totally-complete if there are no covers of* A *of the form* $\{L_x^c : x \in D\}$, $D \subseteq A$, *with* D *totally ordered.*
(b) *Let* E *be a topological space. We say that* A *is* τ*-order-totally-complete if there are no covers of* A *of the form* $\{(\mathrm{cl}\, L_x)^c : x \in D\}$, $D \subseteq A$, *with* D *totally ordered.*

The order-total-completeness admits some equivalent formulations. One of them is related to property (Z) introduced in [18]: a set A has such a property if every totally ordered subset of A has a lower bound in A. The proof of the next proposition follow from Definition 2.2(a).

Proposition 2.3 *The following statements are equivalent:*

(a) *A is order-totally-complete;*
(b) *A has property (Z);*
(c) *every maximal totally ordered subset of A has a lower bound in A.*

Part (a) of the next definition was used in [18] to establish an interesting existence theorem, which will be obtained as a consequence of Theorem 3.4, whereas Part (b) is new.

Definition 2.4

(a) [18] *We say that* A *is order-semicompact (respectively order-s-semicompact) if every cover of* A *of the form* $\{L_x^c : x \in D\}$, $D \subseteq A$ *(respectively* $D \subseteq E$*), has a finite subcover.*
(b) *Let* E *be a topological space. We say that* A *is* τ*-order-semicompact (respectively* τ*-order-s-semicompact) if every cover of* A *of the form* $\{(\mathrm{cl}\, L_x)^c : x \in D\}$, $D \subseteq A$ *(respectively* $D \subseteq E$*), has a finite subcover.*

Clearly, if A is order-s-semicompact then A is order-semicompact, and τ-order-s-semicompactness implies τ-order-semicompactness.

We recall that a directed set $(I, >)$ is a nonempty set I together with a reflexive and transitive relation $>$ such that for any two elements $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\gamma > \alpha$ and $\gamma > \beta$. A net in E is a map from a directed set $(I, >)$ to E. A net $\{y_\alpha : \alpha \in I\}$ is said to be *decreasing* if $y_\beta \preccurlyeq y_\alpha$ for each $\alpha, \beta \in I$, $\beta > \alpha$.

When E is a topological vector space and the ordering is given by a convex cone K the notion of τ-order-semicompactness coincides with K-semicompactness introduced in Definition 2.5 of [3], which has its origin in [21]. In addition, the notion in Part (a) (respectively Part (b)) of the next definition corresponds to the notion of strongly K-completeness (respectively K-completeness) introduced by Luc in Chapter 2 of [16], see also [17]. Recall that a convex cone K is pointed if $K \cap (-K) = \{0\}$.

Definition 2.5

(a) *We say that A is order-complete if there are no covers of the form $\{L^c_{x_\alpha} : \alpha \in I\}$ where $\{x_\alpha : \alpha \in I\}$ is a decreasing net in A.*
(b) *Let E be a topological space. We say that A is τ-order-complete if there are no covers of the form $\{(\mathrm{cl}\, L_{x_\alpha})^c : \alpha \in I\}$ where $\{x_\alpha : \alpha \in I\}$ is a decreasing net in A.*

The next theorem is easy to check and shows the generality of our notions introduced in Definitions 2.2, 2.4 and 2.5.

Theorem 2.6

(a) *If A is order-semicompact then A is order-complete.*
(b) *If A is order-complete then A is order-totally-complete.*
(c) *Let E be a topological space.*

If A is τ-order-semicompact then A is τ-order-complete.
If A is τ-order-complete then A is τ-order-totally-complete.
If A is order-complete then A is τ-order-complete.
If A is order-totally-complete then A is τ-order-totally-complete.

Consequently, we have the following relationships:

$$\begin{array}{ccc} \text{order-}s\text{-semicompact} & & \tau\text{-order-}s\text{-semicompact} \\ \Downarrow & & \Downarrow \\ \text{order-semicompact} & & \tau\text{-order-semicompact} \\ \Downarrow & & \Downarrow \\ \text{order-complete} & \Rightarrow & \tau\text{-order-complete} \\ \Downarrow & & \Downarrow \\ \text{order-totally-complete} & \Rightarrow & \tau\text{-order-totally-complete} \end{array}$$

It is easy to prove the following proposition.

Proposition 2.7

(a) *If A is order-s-semicompact then A_x is order-s-semicompact for each $x \in A$.*
(b) *If A is order-semicompact then A_x is order-semicompact for each $x \in A$.*
(c) *If A is order-complete then A_x is order-complete for each $x \in A$;*
(d) *If A is order-totally-complete then A_x is order-totally-complete for each $x \in A$.*

Proposition 2.8 *Assume that $a \in \mathrm{Min}(A, \preccurlyeq)$. Then,*

(a) $A_a = \{x \in A : \ x \preccurlyeq a, \ a \preccurlyeq x\}$;
(b) *A_a is order-semicompact.*

Proof Part (a) is straightforward. We only prove Part (b). Suppose that

$$A_a \subseteq \bigcup_{x \in D} L^c_x$$

for $D \subseteq A_a$. By (a), it is easy to check that $L_x = L_a$ for all $x \in D$. Thus, $A_a \subseteq L^c_a$, which is a contradiction. Hence A_a is order-semicompact. □

Note that if $a \in \mathrm{Min}(A, \preccurlyeq)$ then A_a is also order-complete or, equivalently, order-totally-complete according to Theorem 2.6.

3 Characterizing the efficiency

In this section, we extend, generalize and in some situations improve some of the more important existence results appearing in the literature. Among them, we mention the results of [16] and [17] obtained in the vectorial case, and also the main result of [18] obtained without linear structure. Our first existence result is without linear and topological structures. Afterwards, we establish several characterizations of the nonemptiness of $\mathrm{Min}(A, \preccurlyeq)$ in the same framework. The case when a topology is added, is also discussed in detail.

When no confusion arises, we use the notation Min A instead of $\mathrm{Min}(A, \preccurlyeq)$.

Next result can be obtained as a consequence of Proposition 2.1(ii) in [19]. However we propose an alternative proof based on our own preliminaries.

Theorem 3.1 *If A is an order-totally-complete set then* Min $A \neq \emptyset$.

Proof Let $\mathcal{P}$ be the set of totally ordered sets in A. Since $A \neq \emptyset$, $\mathcal{P} \neq \emptyset$. Moreover, $\mathcal{P}$ equipped with the partial order given by the inclusion, becomes a partially ordered set. By standard arguments we can prove that any chain in $\mathcal{P}$ has an upper bound and, by Zorn's lemma, we get a maximal set $D \in \mathcal{P}$.

Applying Proposition 2.3, there exists a lower bound $a \in A$ of D. We claim that $a \in \mathrm{Min}\, A$. Indeed, if $a' \in A$ satisfies that $a' \preccurlyeq a$ then a' is also a lower bounded of D. Thus, $a' \in D$ by the maximality of D in $\mathcal{P}$. Hence, $a \preccurlyeq a'$ and therefore $a \in \mathrm{Min}\, A$. □

In particular, by Theorem 2.6, we deduce that if $A \subseteq E$ is order-s-semicompact, order-semicompact or order-complete then Min $A \neq \emptyset$.

By considering E to be a topological space and A to be a τ-order-complete or a τ-order-totally-complete set, we can obtain a sufficient condition for the existence of efficient points under an additional assumption on the partial order. Such an assumption is a topological version of the notion of correcteness introduced in [16], see also [17].

Theorem 3.2 *Let E be a topological space. Suppose that the following condition holds*

$$y \in \mathrm{cl}\, L_x, \quad z \in L_y \cap S_y^c \quad \Longrightarrow \quad z \in L_x. \tag{1}$$

If A is a τ-order-totally-complete set then Min $A \neq \emptyset$.

Proof Let $\mathcal{P}$ be the set of totally ordered sets in A. Obviously $\mathcal{P} \neq \emptyset$. We consider the partial order given by the inclusion on $\mathcal{P}$. By standard arguments we can prove that any chain in $\mathcal{P}$ has an upper bound and, by the Zorn lemma, we get a maximal element in $\mathcal{P}$, say $D \in \mathcal{P}$.

Suppose that Min $A = \emptyset$. We will prove that $\{(\mathrm{cl}\, L_d)^c : d \in D\}$ is a cover of A, giving a contradiction. Indeed, if $y \in A$ and $y \notin \bigcup\{(\mathrm{cl}\, L_d)^c : d \in D\}$ then

$$y \in \mathrm{cl}\, L_d \ \ \forall\, d \in D. \tag{2}$$

Since Min $A = \emptyset$, then there exists $y' \in A$ such that $y' \preccurlyeq y$ and $y \not\preccurlyeq y'$, which means $y' \in L_y \cap S_y^c$. From this and (2), by (1) we deduce that $y' \preccurlyeq d \quad \forall\, d \in D$. The maximality of D implies that $y' \in D$. Thus, D is maximal and bounded below by y'. Hence $y' \in \mathrm{Min}\, A$ which is a contradiction. □

Note that if L_y is closed for all $y \in E$ then condition (1) trivially holds. In particular, if the ordering comes from a convex closed cone K, the following result is a consequence of the preceding theorem. See also [19,Theorem 5.1].

Corollary 3.3 *[3,Theorem 3.1] Suppose that E is a topological vector space ordered by a convex cone K. If A is a τ-order-semicompact set with respect to $\preccurlyeq_K$, then* $\mathrm{Min}(A, \preccurlyeq_{\mathrm{cl}\,K})$ *is nonempty. If, in addition,* $\mathrm{cl}\,K$ *is pointed then* $\mathrm{Min}(A, \preccurlyeq_K)$ *is nonempty.*

Proof Since A is also τ-order-semicompact with respect to $\preccurlyeq_{\mathrm{cl}\,K}$, Theorem 3.2 implies that $\mathrm{Min}(A, \preccurlyeq_{\mathrm{cl}\,K})$ is nonempty. The remaining conclusion follows from the inclusion

$$\mathrm{Min}(A, \preccurlyeq_{\mathrm{cl}\,K}) \subseteq \mathrm{Min}(A, \preccurlyeq_K)$$

which is a consequence of the pointedness of $\mathrm{cl}\,K$. □

Theorem 3.4 *The following assertions are equivalent:*

(a) $\mathrm{Min}\,A \neq \emptyset$*;*
(b) *A has a maximal totally ordered subset minorized by an order-s-semicompact subset H of S_A;*
(c) *A has a nonempty section which is order-complete;*
(d) *A has a nonempty section which is order-totally-complete.*

Proof (a) $\Longrightarrow$ (b): Take any $a \in \mathrm{Min}\,A$, and consider

$$\mathcal{P} \doteq \{D \subseteq E : L_a \cap A \subseteq D \subseteq S_a \cap A \text{ and } D \text{ is totally ordered }\}.$$

Taking into account Proposition 2.8 and $L_a \cap A$ is totally ordered, $L_a \cap A \in \mathcal{P}$. By equipping $\mathcal{P}$ with the partial order given by the inclusion, we can prove by standard arguments that any chain in $\mathcal{P}$ has an upper bound. Therefore, there exists a maximal totally ordered element $D_0 \in \mathcal{P}$, that is,

$$L_a \cap A \subseteq D_0 \subseteq S_a \cap A \subseteq S_a.$$

Set $H = \{a\}$. Then D_0 is minorized by H which is obviously an order-s-semicompact subset of S_A.
(b) $\Longrightarrow$ (c): Applying Theorem 2.1 in [18] we have that $\mathrm{Min}\,A \neq \emptyset$. Take any $a \in \mathrm{Min}\,A$. Then by Proposition 2.8(b), the section A_a is order-complete.
(c) $\Longrightarrow$ (d): It is straightforward.
(d) $\Longrightarrow$ (a): It is a consequence of Theorem 3.1 and Proposition 2.1(b). □

We point out that the previous theorem generalizes [18,Theorem 2.1] and Theorem 3.4 of [16,Chapter 2] (see also Remark 2.14 in [17]). In fact: in [18] is only proven that (b) implies (a); when E is a vector space ordered by a convex cone, the equivalence between (a) and (c) is Theorem 3.4 in [16,Chapter 2] due to Theorem 2.6.

If order-s-semicompactness for the subset H of S_A in Part (b) is replaced by order-semicompactness, then the above theorem may be not true as the following example shows.

Example 3.5 Consider $E = \mathbb{R}^2$ ordered by $K = \mathbb{R}^2_+$ and $A = \{(x + \frac{1}{n}, -x + \frac{1}{n}) : n \in \mathbb{N},\ x \in [0, +\infty[\}$. Set $H = \{t(\frac{1}{2}, -\frac{1}{4}) : \ 0 < t \leq 1\}$. It is easy to check that $H \subseteq S_A$ and $\{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a maximal totally ordered subset of A minorized by H. In addition, H is order-semicompact but is not order-s-semicompact. However, $\mathrm{Min}\,A = \emptyset$.

By virtue of the previous example, it is possible to impose a condition weaker than the order-s-semicompactness for the subset H of S_A, but we have to require an additional property on H. More precisely, we obtain the following theorem.

Theorem 3.6 *The set* Min A *is not empty if and only if* A *has a maximal totally ordered subset* D *minorized by an order-totally-complete subset* H *of* S_A *such that for each* $d \in D$ *there exists* $v \in H$ *such that* $v \sim d$.

Proof Let us see that D has a lower bound in A. Then, as a consequence Min $A \neq \emptyset$. On the contrary, suppose that for each $a \in A$ there is $d \in D$ such that $a \in L_d^c$. Then we have

$$A \subseteq \bigcup_{d \in D} L_d^c.$$

Thus, $S_A \subseteq \bigcup_{d \in D} L_d^c$ and

$$H \subseteq \bigcup_{d \in D} L_d^c. \tag{3}$$

By hypothesis, for each $d \in D$ there is $v \in H$ with $v \sim d$, taking into account (3) and $L_d = L_v$ we can find a totally ordered subset H' of H such that

$$H \subseteq \bigcup_{v \in H'} L_v^c$$

which contradicts that H is order-totally-complete.

Suppose that $a \in$ Min A. Consider

$$\mathcal{P} \doteq \{P \subseteq E \colon L_a \cap A \subseteq P \subseteq A \text{ and } P \text{ is totally ordered}\}.$$

It is clear that, $L_a \cap A \in \mathcal{P}$ (see Proposition 2.8). By equipping $\mathcal{P}$ with the partial order given by the inclusion, we can prove by standard arguments that there exists a maximal totally ordered element $P_0 \in \mathcal{P}$. Moreover, by Proposition 2.3, P_0 is order-totally-complete. Hence, if $D = H = P_0$ we conclude the proof. □

In the context of topological spaces we obtain the following characterization of efficiency which is new in this general setting and generalizes Theorem 3.3 in [16,Chapter 2] (see also [17,Theorem 2.6]).

Theorem 3.7 *Suppose that* E *is a topological space and condition* (1) *holds. Then, the following assertions are equivalent:*

(a) Min $A \neq \emptyset$;
(b) *A has a nonempty section which is τ-order-complete;*
(c) *A has a nonempty section which is τ-order-totally-complete.*

Proof (a) $\Longrightarrow$ (b): Take $a \in$ Min A and consider the section A_a. Then the result follows from Proposition 2.8.
(b) $\Longrightarrow$ (c): It is immediate.
(c) $\Longrightarrow$ (a): It follows from Theorem 3.2 and Proposition 2.1(b). □

4 Characterizing the domination property

In some situations arising in the theory of decision making, we are interested in the subset of efficient points for which no other admissible (alternative) point is preferred. We say that an admissible set has the domination property is each admissible point is either efficient, or else there exists a preferred admissible point which is also efficient. It is mathematically expressed in the following definition.

Definition 4.1 *A set A has the domination property if*

$$\text{for each } a \in A \text{ there exists } a_0 \in \operatorname{Min} A \text{ such that } a_0 \in L_a \cap A,$$

or equivalently,

$$A \subseteq S_{\operatorname{Min} A}.$$

As a consequence, A has the domination property if and only if

$$\operatorname{Min}(L_a \cap A) \neq \emptyset \qquad \forall\, a \in A.$$

Due to Proposition 2.7 and Theorems 2.6 and 3.1 we have that if A is order-s-semicompact, order-semicompact, order-complete or order-totally-complete then A has the domination property.

The following characterization holds.

Theorem 4.2 *The following statements are equivalent:*

(a) *A has the domination property;*
(b) *for each $a \in A$, there exists $a_0 \in L_a \cap A$ such that $L_{a_0} \cap A$ has a maximal totally ordered subset minorized by an order-s-semicompact subset of $S_{L_{a_0} \cap A}$;*
(c) *for each $a \in A$ there exists $a_0 \in L_a \cap A$ such that $L_{a_0} \cap A$ is order-complete;*
(d) *for each $a \in A$ there exists $a_0 \in L_a \cap A$ such that $L_{a_0} \cap A$ is order-totally-complete.*

Proof $(a) \Longrightarrow (b)$: Let $a \in A$. By assumption, we can take $a_0 \in \operatorname{Min}(L_a \cap A)$. Set

$$D \doteq (L_a \cap A) \cap L_{a_0} = L_{a_0} \cap A = \{x \in A : \; x \preccurlyeq a_0, \; a_0 \preccurlyeq x\}.$$

Obviously D is totally ordered, is maximal in $L_{a_0} \cap A$, and it is minorized by $H \doteq \{a_0\}$, which is order-s-semicompact.
$(b) \Longrightarrow (c)$: By Theorem 2.1 [18], there exists $a' \in \operatorname{Min}(L_{a_0} \cap A)$. Thus, by Proposition 2.8(b), $A_{a'} = L_{a'} \cap A$ is order-complete.
$(c) \Longrightarrow (d)$. It is immediate.
$(d) \Longrightarrow (a)$. It is follows from Theorem 3.1 and Proposition 2.1(b). □

Similarly, taking into account Proposition 2.8 and Theorem 3.2 we have the following characterization.

Theorem 4.3 *Suppose that E is a topological space and the condition* (1) *holds. The following statements are equivalent:*

(a) *A has the domination property;*
(b) *for each $a \in A$ there exists $a_0 \in L_a \cap A$ such that $L_{a_0} \cap A$ is τ-order-complete;*
(c) *for each $a \in A$ there exists $a_0 \in L_a \cap A$ such that $L_{a_0} \cap A$ is τ-order-totally-complete.*

5 Optimization problems

In the sequel, X is a Hausdorff topological space and E is a nonempty set equipped with a partial order $\preccurlyeq$ and f is a map from X into E.

Consider the following abstract optimization problem

$$(P) \qquad \operatorname{Min} f(x) \text{ subject to } x \in X,$$

We set $f(X) \doteq \{f(x)\colon x \in X\}$. By a solution to (P), we mean a point $\bar{x} \in X$ such that $f(\bar{x}) \in \mathrm{Min}(f(X), \preccurlyeq)$. Thus, our main goal in this section is stating some conditions guaranteeing $\mathrm{Min}\, f(X) \doteq \mathrm{Min}(f(X), \preccurlyeq) \neq \emptyset$. Actually, our conditions will imply the domination property. These results will be specialized to vector optimization problems, that is, when the ordering is induced by a convex cone.

Theorem 5.1 *Let X be compact. If $f^{-1}(L_y)$ is closed for each $y \in f(X)$(resp. for each $y \in E$), then*

(a) $f(X)$ is order-semicompact (resp. $f(X)$ is order-s-semicompact);
(b) $f(X)$ has the domination property, that is, every lower section of $F(X)$ has an efficient point. In particular, $\mathrm{Min}\, f(X) \neq \emptyset$.

Proof We only prove Part (a) when $f^{-1}(L_y)$ is closed for each $y \in f(X)$. Part (b) follows from Proposition 2.7(b), Theorems 2.6(a)-(b) and 3.1. Suppose that $\bigcup_{d \in D} L_d^c$ is a cover of $f(X)$ with $D \subseteq f(X)$. Set

$$U_d \doteq \{x \in X\colon f(x) \notin L_d\}.$$

Thus, $X = \bigcup_{d \in D} U_d$. Since $f^{-1}(L_d)$ is closed and $U_d = (f^{-1}(L_d))^c$, then U_d is open for each $d \in D$. Moreover, as X is compact there exists a finite set $\{d_1, \ldots, d_r\} \subseteq D$ such that

$$X = U_{d_1} \cup \cdots \cup U_{d_r}.$$

Hence, $L_{d_1}^c \cup \cdots \cup L_{d_r}^c$ covers $f(X)$ and $f(X)$ is order-semicompact. □

The previous theorem admits a variant when the closedness of $f^{-1}(\mathrm{cl}\, L_y)$ for all $y \in E$ is required. In this case, we need an additional assumption on the partial order. Notice that in this case the domination property is not obtained.

Theorem 5.2 *Suppose that E is a topological space and the condition* (1) *holds. Let X be compact. If $f^{-1}(\mathrm{cl}\, L_y)$ is closed for each $y \in f(X)$ (resp. for each $y \in E$), then*

(a) $f(X)$ is τ-order-semicompact (resp. $f(X)$ is τ-order-s-semicompact),
(b) $\mathrm{Min}\, f(X) \neq \emptyset$.

Proof We can apply the same arguments as in Theorem 5.1 by considering

$$U_d = \{x \in X\colon f(x) \notin \mathrm{cl}\, L_d\}$$

and taking into account Theorem 3.2. □

When E is a topological vector space and $\preccurlyeq = \preccurlyeq_K$ for some convex cone K, a function $f: X \to E$ is said to be K-semicontinuous ([3]) if $f^{-1}(\mathrm{cl}(L_y))$ is closed for all $y \in E$, i.e., if $f^{-1}(y - \mathrm{cl}(K))$ is closed for all $y \in E$. We recall that K-semicompactness [3] coincides with the the notion of τ-order-semicompactness with respect to $\preccurlyeq_K$, as already mentioned before Definition 2.4.

Corollary 5.3 *Suppose that E is a topological vector space ordered by a convex cone K. Let X be compact. If f is K-semicontinuous and K is closed, then*

(a) $f(X)$ is order-s-semicompact. In particular, $f(X)$ is K-semicompact.
(b) $f(X)$ has the domination property,
(c) $\mathrm{Min}\, f(X) \neq \emptyset$.

Corollary 5.4 *[3,Corollary 3.1] Suppose that E is a topological vector space ordered by a convex cone K. Let X be compact. If f is K-semicontinuous and* $\mathrm{cl}\,K$ *is pointed, then* $\mathrm{Min}\, f(X) \neq \emptyset$.

Proof It is a consequence of Corollaries 5.3 and 3.5 by noticing that K-semicontinuity is equivalent to $\mathrm{cl}(K)$-semicontinuity. □

We now introduce the following relaxed continuity for single valued maps.

Definition 5.5 *Let $x_0 \in X$. We say that f is (resp. τ-)decreasingly lower bounded at x_0 if for each net $\{x_\alpha : \alpha \in I\}$ converging to x_0 such that $\{f(x_\alpha) : \alpha \in I\}$ is decreasing, the following holds*

$$\exists\, \alpha_0 \in I \;\; \forall\, \alpha > \alpha_0 : \quad f(x_0) \in L_{f(x_\alpha)} \quad (\text{resp. } \; f(x_0) \in \mathrm{cl}\, L_{f(x_\alpha)}),$$

or equivalently (because $(f(x_\alpha))_{\alpha \in I}$ is decreasing and I is a direct set),

$$\forall\, \alpha \in I : \quad f(x_0) \in L_{f(x_\alpha)} \quad (\text{resp. } \; f(x_0) \in \mathrm{cl}\, L_{f(x_\alpha)}).$$

We say that f is (resp. τ-)decreasingly lower bounded (on X) if it is at every $x_0 \in X$.

Proposition 5.6 *If $f^{-1}(L_y)$ is closed (resp. $f^{-1}(\mathrm{cl}\, L_y)$) for each $y \in f(X)$, then f is (resp. τ-) decreasingly lower bounded.*

Proof Suppose that $\{x_\alpha : \alpha \in I\}$ is a net in X, converging to $x_0 \in X$ and $\{f(x_\alpha) : \alpha \in I\}$ is decreasing. Let us prove that $f(x_0) \in L_{f(x_\alpha)}$ for all $\alpha \in I$. If there exists $\alpha' \in I$ such that $f(x_0) \not\preccurlyeq f(x_{\alpha'})$, then $f(x_0) \not\preccurlyeq f(x_\alpha)$ for each $\alpha \in I$ with $\alpha > \alpha'$ since $\{f(x_\alpha) : \alpha \in I\}$ is decreasing. Thus, $\{x_\alpha : \alpha \in I,\ \alpha > \alpha'\} \subseteq f^{-1}(L_{f(x_{\alpha'})})$. Therefore, since $f^{-1}(L_{f(x_{\alpha'})})$ is closed we have $x_0 \in f^{-1}(L_{f(x_{\alpha'})})$. Hence, $f(x_0) \preccurlyeq f(x_{\alpha'})$ which is a contradiction.
We can reason in a similar way to check that f is τ-decreasingly lower bounded when $f^{-1}(\mathrm{cl}\, L_y)$ is closed for each $y \in f(X)$. □

Remark 5.7 As a consequence of Proposition 5.6, if E is a topological vector space ordered by a closed convex cone K and f is K-semicontinuous then f is decreasingly lower bounded.

Theorem 5.8 *Let X be compact. If f is decreasingly lower bounded, then*

(*a*) *$f(X)$ is order-complete,*
(*b*) *$f(X)$ has the domination property,*
(*c*) $\mathrm{Min}\, f(X) \neq \emptyset$.

Proof Suppose that $\bigcup_{\alpha \in I} L^c_{y_\alpha}$ is a cover of $f(X)$ with $\{y_\alpha : \alpha \in I\}$ a decreasing net in $f(X)$. Let x_α be such that $f(x_\alpha) = y_\alpha$ for each $\alpha \in I$. Since X is compact there exists a subnet $\{x_{\alpha'} : \alpha \in I'\}$ of $\{x_\alpha : \alpha \in I\}$ such that converges to some $x_0 \in X$. Therefore exists $\alpha'_0 \in I'$ such that

$$f(x_0) \in L_{f(x_\alpha)} \quad \forall\, \alpha' \in I,\ \ \alpha' > \alpha'_0. \tag{4}$$

On the other hand, $f(x_0) \in L^c_{y_{\alpha_0}}$ for some $\alpha_0 \in I$. Let $\gamma \in I'$ such that $\gamma > \alpha_0$ and $\gamma > \alpha'_0$. Since $\{y_\alpha : \alpha \in I\}$ is decreasing we have $L^c_{y_{\alpha_0}} \subseteq L^c_{y_\gamma}$. Thus,

$$f(x_0) \in L^c_\gamma$$

which contradicts (4). Consequently, $f(X)$ is order-complete and the proof is concluded. □

We note that similar arguments prove that if f is τ-decreasingly lower bounded then $f(X)$ is τ-order-complete. In addition, if condition (1) is imposed, then $\mathrm{Min}\, f(X) \neq \emptyset$ by Theorem 3.2.

6 Application to set-valued optimization: the set criterion case

This section is devoted to set-valued optimization. As an application to our results we extend and improve several existence theorems given in [14,15] and [1] and present new ones as well.

Throughout this section Y will be a topological vector space ordered by a convex closed cone $K \subseteq Y$ and consider the following set relations $\preccurlyeq^l$ and $\preccurlyeq^u$ defined between nonempty subsets of Y. If $A, B \in \wp_0(Y)$ then

$$A \preccurlyeq^l B \quad \text{if and only if} \quad B \subseteq A + K.$$

$$A \preccurlyeq^u B \quad \text{if and only if} \quad A \subseteq B - K.$$

Such relations $\preccurlyeq^l$ and $\preccurlyeq^u$ are reflexive and transitive on $\wp_0(Y)$. Moreover, since $A \preccurlyeq^l B \Leftrightarrow B + K \subseteq A + K$, it is clear that $\preccurlyeq^l$ can not be antisimetric, unless $K = \{0\}$.

Note that $A \preccurlyeq^u B \Leftrightarrow (-A) \subseteq (-B) + K \Leftrightarrow (-B) \preccurlyeq^l (-A)$ therefore, there is no need to consider both relations.

Kuroiwa in [13] introduced the notion of efficient set for a family $\mathcal{F} \subseteq \wp_0(Y)$ as follows. It is said that $A \in \mathcal{F}$ is an l-minimal (u-minimal) set, and we write $A \in l\text{-Min}\,\mathcal{F}$ ($A \in u\text{-Min}\,\mathcal{F}$) if $B \in \mathcal{F}$ and $B \preccurlyeq^l A$ ($B \preccurlyeq^u A$) imply that $A \preccurlyeq^l B$ ($A \preccurlyeq^u B$).

In this manner, $(\wp_0(Y), \preccurlyeq^l)$ and $(\wp_0(Y), \preccurlyeq^u)$ are particular cases of $(E, \preccurlyeq)$ studied in the preceding sections. So, taking into account the notations and definitions introduced in Sect. 2, we obtain the corresponding concepts for families of sets as follows.

Let $B \in \wp_0(Y)$, associated to the ordering $\preccurlyeq^l$, the lower section at B, is

$$L_{(B,\preccurlyeq^l)} = \{A \in \wp_0(Y) \colon A \preccurlyeq^l B\} = \{A \in \wp_0(Y) \colon B \subseteq A + K\}$$

and its complement

$$\begin{aligned} L^c_{(B,\preccurlyeq^l)} \doteq (L_{(B,\preccurlyeq^l)})^c &= \{A \in \wp_0(Y) \colon A \not\preccurlyeq^l B\} \\ &= \{A \in \wp_0(Y) \colon B \nsubseteq A + K\} \\ &= \{A \in \wp_0(Y) \colon \exists\, b \in B,\ A \subseteq (b - K)^c\}. \end{aligned}$$

Since we want to compare with most of the results appearing in the literature, the set-relation $\preccurlyeq^u$ is also considered explicitely just for convenience of the reader. We have

$$L_{(B,\preccurlyeq^u)} = \{A \in \wp_0(Y) \colon A \preccurlyeq^u B\} = \{A \in \wp_0(Y) \colon A \subseteq B - K\}$$

and

$$\begin{aligned} L^c_{(B,\preccurlyeq^u)} \doteq (L_{(B,\preccurlyeq^u)})^c &= \{A \in \wp_0(Y) \colon A \not\preccurlyeq^u B\} \\ &= \{A \in \wp_0(Y) \colon A \cap (B - K)^c \neq \emptyset\} \\ &= \{A \in \wp_0(Y) \colon \exists\, a \in A,\ B \subseteq (a + K)^c\} \end{aligned}$$

Then, by Definitions 2.2, 2.4 and 2.5(a) we obtain the notion of family order-s-semicompact, order-semicompact, order-complete and order-totally-complete with respect to $\preccurlyeq^l$ or $\preccurlyeq^u$. More precisely, given $\mathcal{F} \subseteq \wp_0(Y)$ and setting $H \doteq \bigcup_{A \in \mathcal{F}} A$, we obtain that $\{L^c_{(B,\preccurlyeq^l)} \colon B \in \mathcal{F}'\}$ is a cover of $\mathcal{F}$ for some $\mathcal{F}' \subseteq \mathcal{F}$ if, and only if, there exists $H' \subseteq H$ such that the family of sets $\{(y - K)^c \colon y \in H'\}$ satisfies the following:

$$\forall\, A \in \mathcal{F},\ \exists\, y \in H',\ A \subseteq (y - K)^c. \tag{5}$$

The family $\{(y-K)^c\colon y \in H'\}$ satisfying the previous property is termed in [1,Definition 27] a $K_{\mathcal{F}}$-cover. In the same paper, it is said that $\mathcal{F}$ is K-semicompact if each $K_{\mathcal{F}}$-cover admits a finite $K_{\mathcal{F}}$-subcover. One can easily prove that K-semicompactness implies order-semicompactness with respect to $\preccurlyeq^l$ but the converse may be not true as the following example shows.

Example 6.1 Consider $\mathbb{R}^2$ ordered by $K = \mathbb{R}^2_+$ and the family $\mathcal{F} = \{A_x\colon x \in]0,+\infty[\}$ defined by $A_x = [(-x,0),(-x,-\frac{1}{x})] \cup \{(\frac{1}{x},-\frac{1}{x})\}$, $x \in]0,+\infty[$. It is easy to check that $\mathcal{F}$ is order-semicompact with respect to $\preccurlyeq^l$. However $\mathcal{F}$ is not K-semicompact since $\{(a_x-K)^c\colon x \in]0,+\infty[\}$ with $a_x = (\frac{1}{x},-\frac{1}{x})$ is a $K_{\mathcal{F}}$-cover but it does not admit any finite $K_{\mathcal{F}}$-subcover.

As a consequence, if $\mathcal{F}$ is K-semicompact then we obtain Proposition 30 in [1] by Theorem 3.1. Moreover, we also proved that $\mathcal{F}$ has the domination property.

Concerning the relation $\preccurlyeq^u$, given $\mathcal{F}' \subseteq \mathcal{F}$, we obtain that $\{L^c_{(B,\preccurlyeq^u)}\colon B \in \mathcal{F}'\}$ covers $\mathcal{F}$ if, and only if, the family $\{(B-K)^c\colon B \in \mathcal{F}'\}$ satisfies the property:

$$\forall\, A \in \mathcal{F},\ \exists\, B \in \mathcal{F}',\ A \cap (B-K)^c \neq \emptyset.$$

Such a family is termed in [1,Definition 19] a K-family of intersection of $\mathcal{F}$. In the same paper, it is said that $\mathcal{F}$ is K-regular if each K-family of intersection of $\mathcal{F}$ admits a finite K-family of intersection. In this case, K-regularity amounts to saying order-semicompactness with respect to $\preccurlyeq^u$. Thus, the existence result of Proposition 22 in [1] is a consequence of our general Theorem 3.1. Also the domination property is satisfied.

Note that Theorem 3.2 in [15] is a consequence of Theorem 3.4.

On the other hand, since the order-completeness with respect to $\preccurlyeq^l$ coincides with the strongly K-completeness defined in [11,Definition 3.9], Theorem 3.1 generalizes [11,Theorem 4.1].

We now consider the following set-valued optimization problem

$$(SP) \qquad \text{Min } F(x) \text{ subject to } x \in X,$$

where X is a Hausdorff topological space and $F\colon X \to \wp_0(Y)$ is a set-valued map. Let us consider $\mathcal{F} \doteq \{F(x)\colon x \in X\}$. We say that $x_0 \in X$ is an l-solution of (SP) if $F(x_0) \in \text{Min}(\mathcal{F},\preccurlyeq^l)$.

Definition 6.2 *[16,Chapter 1] We say that F is upper K-semicontinuous at $x_0 \in X$ if for each open neighborhood V of $F(x_0)$ there is an open neighborhood U of x_0 in X such that*

$$F(x) \subseteq V + K \qquad \forall\, x \in U.$$

We say that F is upper K-semicontinuous on X if it is at every $x \in X$.

Proposition 6.3 *Assume that F is upper K-semicontinuous on X. Then*

(*a*) $F^{-1}(L_{(F(x),\preccurlyeq^l)}) \doteq \{x' \in X\colon F(x') \preccurlyeq^l F(x)\}$ *is closed for any* $x \in X$;
(*b*) *if, in addition, X is compact, every $K_{\mathcal{F}}$-cover of the form*

$$\{(y-K)^c\colon y \in H'\}, \quad H' \subseteq \bigcup_{x\in X} F(x)$$

(*in the sense of* (5)) *admits a finite $K_{\mathcal{F}}$-subcover.*

Proof
(a): Fix $\overline{x} \in X$ and take $x \in X$ such that $F(x) \not\preccurlyeq^l F(\overline{x})$. This means that there exists $\overline{y} \in F(\overline{x})$ with $\overline{y} \notin F(x) + K$, that is, $F(x) \subseteq (\overline{y} - K)^c$. Since K is closed and F is upper K-semicontinuous, there exists U neighborhood of x such that $F(x') \subseteq (\overline{y} - K)^c + K = (\overline{y} - K)^c$ for every $x' \in U$. Hence $\overline{y} \notin F(x') + K$, whence $F(x') \not\preccurlyeq^l F(\overline{x})$, for every $x' \in U$.
(b): This is Proposition 29 in [1]. □

We will show that the compactness of X cannot be remove in (b) of the previous proposition. Indeed, take $F : X \to \wp_0(\mathbb{R}^2)$, $F(\lambda) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \lambda^2\}$, $X =]0, 1[$, with $\mathbb{R}^2$ ordered by $K = \mathbb{R}^2_+$. Then, F is K-semicontinuous on X. However, $\{((-\lambda, 0) - K)^c$: $\lambda \in]0, 1[\}$ is a $K_{\mathcal{F}}$-cover that admits no finite $K_{\mathcal{F}}$-subcover.

From (a) of the previous proposition and Theorem 5.1 we obtain the following existence theorem. Nevertheless, by (b) of the same proposition a stronger property is obtained for $\mathcal{F}$ in [1,Proposition 29], see also [11,Theorem 5.3].

Theorem 6.4 *Suppose that X is compact and F is upper K-semicontinuous on X. Then $\mathcal{F}$ is order-semicompact with respect to $\preccurlyeq^l$, has the domination property and problem (SP) has an l-solution.*

The next proposition shows the importance of our notion introduced in Definition 5.5.

Let $\{A_\alpha : \alpha \in I\}$ be a net of subsets of Y. We denote $\text{Limsup}_\alpha A_\alpha$ the set of all cluster points of $\{y_\alpha : y_\alpha \in A_\alpha, \alpha \in I\}$.

Proposition 6.5 *Let $x_0 \in X$. If F is l-type demi-lower semicontinuous in the sense of Kuroiwa [14,Definition 3.2] at x_0 then F is decreasingly lower bounded at x_0.*

Proof Suppose that $\{x_\alpha\} \subseteq X$ converges to x_0 such that $\{F(x_\alpha) : \alpha \in I\}$ is decreasing with respect to $\preccurlyeq^l$. This implies

$$\bigcup_{\alpha \in I} F(x_\alpha) \subseteq \text{Limsup}_\alpha (F(x_\alpha) + K).$$

By assumption, F is l-type demi-lower semicontinuous at x_0, that is,

$$F(x_0) \preccurlyeq^l \text{Limsup}_\alpha (F(x_\alpha) + K).$$

Therefore, $F(x_0) \preccurlyeq^l F(x_\alpha) \;\; \forall \, \alpha \in I$. □

Thus, [14,Theorem 4.1] and by symmetry [14,Theorem4.3] follow from Theorem 5.8. Moreover, [11,Theorem5.8] is a consequence of Proposition 6.5 and Theorem 5.8.

A more verifiable condition implying the l-type demi-lower semicontinuity of F is the following: for any $x \in X$, $F(x) + K$ is closed and

$$\{x' \in X : \; F(x') \preccurlyeq^l F(x)\} \text{ is closed.}$$

In what follows, given any $A \subseteq Y$, we set

$$F^-(A) \doteq \{x \in X : \; F(x) \cap A \neq \emptyset\}.$$

The following result is easy to check and appears in Remark 2.1 in Ferro [5].

Proposition 6.6 *Assume that F is upper K-semicontinuous on X and K is closed. Then $F^-(L_y)$ is closed for all $y \in Y$.*

When $y \in Y$, it is not difficult to show that

$$F^-(L_y) = \{x \in X : \ F(x) \preccurlyeq^l y\}.$$

Therefore, from the above proposition we obtain that [11,Theorem5.9] is related to Theorem 6.4.

7 Application to set-valued optimization: the vector criterion case

In the sequel, X and E are two Hausdorff topological spaces and Y is equipped with any partial order $\preccurlyeq$. Consider the following set-valued optimization problem

$$(VP) \quad \text{Min } F(x) \text{ subject to } x \in X$$

where F is a set-valued map from X to E such that $F(x) \neq \emptyset$ for each $x \in X$.

It is said that $x_0 \in X$ is a solution of problem (VP) if there exists $y_0 \in F(x_0)$ such that $y_0 \in \text{Min}(F(X), \preccurlyeq)$ where $F(X) = \bigcup_{x \in X} F(x)$. Therefore, solving problem (VP) means to find conditions under which $\text{Min } F(X) \doteq \text{Min}(F(X), \preccurlyeq) \neq \emptyset$.

Definition 7.1 [*18*] *It is said that F is inf-compact if there exists $y_0 \in F(X)$ such that $F^-(L_y)$ is compact for each $y \in L_{y_0}$.*

The next existence theorem is obtained in [18,Corollary 2.2] under the stronger assumption that $F(x)$ is order-s-semicompact for all $x \in X$, since order-s-semicompactness of $F(x)$ implies order-s-semicompactness of $F(x) \cap L_y$ for each $y \in F(X)$, and the converse may be not true. Indeed, consider $\mathbb{R}^2$ ordered by $\mathbb{R}^2_+$ and $F: \mathbb{R} \longrightarrow 2^{\mathbb{R}^2}$ defined by $F(x) = \{(t - x, -t) : t \in]0, 1]\}$ for each $x \in \mathbb{R}$. One can check that none of the image sets $F(x)$ are order-s-semicompact. However for each $y \in F(\mathbb{R})$ and $x \in \mathbb{R}$ the set $F(x) \cap L_y$ is order-s-semicompact or empty.

Theorem 7.2 *Suppose that $F(x) \cap L_y$ is order-s-semicompact for each $x \in X$ and $y \in F(X)$. If F is inf-compact then* $\text{Min } F(X) \neq \emptyset$.

Proof Let $y_0 \in F(X)$ such that $F^-(L_y)$ is compact for each $y \in L_{y_0}$. It is sufficient to check that $F(X) \cap L_{y_0}$ is order-complete. Suppose that $\bigcup_{\alpha \in I} L^c_{d_\alpha}$ is a covering of $F(X) \cap L_{y_0}$ with $\{d_\alpha : \alpha \in I\}$ being a decreasing net in $F(X) \cap L_{y_0}$. For each $\alpha \in I$, we consider the set

$$U_\alpha \doteq \{x \in X : \ F(x) \subseteq L^c_{d_\alpha}\} = \{x \in X : \ F(x) \cap L_{d_\alpha} = \emptyset\} = [F^-(L_{d_\alpha})]^c.$$

Since $d_\alpha \in F(X) \cap L_{y_0}$ and F is inf-compact then U_α is open. Let us prove that

$$F^-(L_{y_0}) \subseteq \bigcup_{\alpha \in I} U_{d_\alpha}. \tag{6}$$

Indeed, if $x \in F^-(L_{y_0})$ then $F(x) \cap L_{y_0} \neq \emptyset$. Since $F(x) \cap L_{y_0} \subseteq F(X) \cap L_{y_0}$ we have

$$F(x) \cap L_{y_0} \subseteq \bigcup_{\alpha \in I} L^c_{d_\alpha}$$

Therefore, by the order-s-semicompactness of $F(x) \cap L_{y_0}$, there exists $d_{\alpha(x)} \in \{d_\alpha : \alpha \in I\}$ such that

$$F(x) \cap L_{y_0} \subseteq L^c_{d_{\alpha(x)}}$$

because of $\{d_\alpha : \alpha \in I\}$ is a decreasing net and since $d_{\alpha(x)} \in L_{y_0}$ we have

$$F(x) = (F(x) \cap L_{y_0}) \cup (F(x) \cap L^c_{y_0}) \subseteq L^c_{d_{\alpha(x)}} \cup L^c_{y_0} \subseteq L^c_{d_{\alpha(x)}}.$$

Consequently, (6) holds. Thus, by the compactness of $F^-(L_{y_0})$ we obtain

$$F^-(L_{y_0}) \subseteq U_{d_\beta} = [F^-(L_{d_\beta})]^c$$

for some $\beta \in I$ with $d_\beta \in F(X) \cap L_{y_0}$, which contradicts the fact that $F^-(L_z) \neq \emptyset$ for all $z \in F(X)$. □

The following notion can be considered an extension of the cone-lower semicontinuity for a set-valued map given by Ferro in [5] and is useful in order to write the inf-compactness assumption on F in terms of more verifiable conditions.

Definition 7.3 *It is said that F is order-lower semicontinuous on X if $F^-(L_y)$ is closed for all $y \in E$.*

Looking at the proof of the previous theorem, we immediately obtain the next corollary. Notice that when X is compact and F is order-lower semicontinuous on X, $F^-(L_y)$ is compact for all $y \in E$. Hence, F is inf-compact.

Corollary 7.4 *Suppose that X is compact, F is order-lower semicontinuous and $F(x) \cap L_y$ is order-s-semicompact for each $x \in X$ and $y \in F(X)$. Then $F(X)$ has the domination property.*

By considering E a topological linear space ordered by a closed convex pointed cone K, Theorem 7.2 and Corollary 7.4 are related to Theorem 3.1(i) and (ii) in [6] respectively where the order-s-semicompantness of $F(x) \cap L_y$ is replaced by the order-completeness of $F(x) + K$.

In the remaining of this section we assume that E is a topological vector space ordered by a convex cone K. Under these assumptions, it is easy to check that $S_{L^c_y} = L^c_y$ and $S_{(\mathrm{cl}\, L_y)^c} = (\mathrm{cl}\, L_y)^c$ for any $y \in E$.

Theorem 7.5 *Let X be compact. Assume that F is upper K-semicontinuous on X, and $F(x)$ is τ-order-s-semicompact for each $x \in X$. The following assertions hold:*

(a) $F(X)$ is τ-order-s-semicompact;
(b) if, in addition, condition (1) *is satisfied, then* $\mathrm{Min}\, F(X) \neq \emptyset$.

Proof As usual we only prove (a). Suppose that $\bigcup_{d \in D} (\mathrm{cl}\, L_d)^c$ is a covering of $F(X)$ with $D \subseteq E$. Then for each $x \in X$ there exists a finite set $J(x)$ in D such that

$$F(x) \subseteq \bigcup_{d \in J(x)} (\mathrm{cl}\, L_d)^c$$

since $F(x)$ is τ-order-s-semicompact. Obviously, the set $V(x) = \bigcup_{d \in J(x)} (\mathrm{cl}\, L_d)^c$ is open. For each $x \in X$ we consider the set

$$U(x) = \{x' \in X : F(x') \subseteq S_{V(x)}\},$$

which is open because of the upper K-semicontinuity of F on X. Thus, $\{U(x) : x \in X\}$ is an open covering of X. Since X is compact, we obtain

$$X \subseteq U(x_1) \cup \cdots \cup U(x_s)$$

where $\{x_1, \ldots, x_s\} \subseteq X$. Consequently, $F(X) \subseteq V(x_1) \cup \cdots \cup V(x_s)$. So, $F(X)$ is τ-order-s-semicompact. Part (b) follows from Theorem 3.2. □

Corollary 7.6 *Let X be compact. Assume that K is closed, F is upper K-semicontinuous on X and $F(x)$ is order-s-semicompact for each $x \in X$. The following assertions hold:*

(a) *$F(X)$ is order-s-semicompact;*
(b) *$F(X)$ has the domination property;*
(c) $\operatorname{Min} F(X) \neq \emptyset$.

We cannot expect that "$F(X)$ is τ-order-semicompact" holds if "$F(x)$ is τ-order-semicompact for each $x \in X$" in Theorem 7.5, as the example in Remark 2.3 of [5] shows (an example when $\preccurlyeq = \preccurlyeq_K$ for some pointed, closed, convex, cone K with non-empty interior, is given). Hence, Lemma 3.1 in [4] is false, and therefore Theorem 3.1 in the same paper remains open. For the same reason, as was pointed out by Ferro, Theorem 5.4 in [16,Chapter 2] should be rewritten. Indeed, Theorem 7.5 and Corollary 7.6 are alternative correct versions of that theorem.

Notice that in Theorem 2.2 in [5] it was imposed the weaker assumption of order-semicompactness for $F(x) + K$ instead of the order-s-semicompactness for $F(x)$ in the previous corollary, but we proved the stronger property of order-s-semicompactness for $F(X)$ in contrast to that in Theorem 2.2 in [5].

Acknowledgements The authors are grateful to the anonymous referees for their helpful comments and suggestions.

References

1. Alonso, M., Rodríguez-Marín, L.: Set-relations and optimality conditions in set-valued maps. Nonlinear Anal. **63**, 1167–1179 (2005)
2. Brink, C.: Power structures. Algebra Universalis. **30**, 177–216 (1993)
3. Corley, H.W.: An existence result for maximizations with respect to cones. J. Optim. Theory Appl. **31**(2), 277–281 (1980)
4. Corley, H.W.: Existence and Lagrangian duality for maximizations of set-valued functions. J. Optim. Theory Appl. **54**(3), 489–501 (1987)
5. Ferro, F.: An optimization result for set-valued mappings and a stability property in vector problems with constraints. J. Optim. Theory Appl. **90**(1), 63–77 (1996)
6. Ferro, F.: Optimization and Stability Results through Cone Lower Semicontinuity. Set-Valued Anal. **5**, 365–375 (1997)
7. Ha, T.X.D.: Some variants of the Ekeland variational principle for a set-valued map. J. Optim. Theory Appl. **124**(1), 187–206 (2005)
8. Hamel, A., Löhne, A.: Minimal set theorems. Report of the Institute of Optimization and Stochastics. (11), (2002)
9. Hartley, R.: On cone-efficiency, cone-convexity and cone-compactness, SIAM. J. Appl. Math. **34**(2), 211–222 (1978)
10. Hernández, E., Rodríguez-Marín, L.: Nonconvex scalarization in set optimization with set-valued maps. J. Math. Anal. Appl. **325**(1), 1–18 (2007)
11. Hernández, E., Rodríguez-Marín, L.: Existence theorems for set optimization problems. Nonlinear Anal. **67**, 1726–1736 (2007)
12. Jahn, J.: Vector optimization, Theory, applications, and extensions. Springer-Verlag, Berlin (2004)
13. Kuroiwa, D.: The natural criteria in set-valued optimization. RIMS Kokyuroku, **1031**, 85–90 (1998)
14. Kuroiwa, D.: Existence theorems of set optimization with set-valued maps. J. Inf. Optim. Sci. **24**(1), 73–84 (2003)
15. Kuroiwa, D.: Existence of efficient points of set optimization with weighted criteria. J. Nonlinear Convex Anal. **4**(1), 117–123 (2003)
16. Luc, D.T.: Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems, 319, Springer-Verlag, Berlin (1989)
17. Luc, D.T.: An existence theorem in vector optimization. Mathematics of Operations Research. **14**(4), 693–699 (1989)

18. Ng, K.F., Zheng, X.Y.: Existence of efficient Points in vector optimization and generalized Bishop-Phelps Theorem. J. Optim. Theory Appl. **115**(1), 29–47 (2002)
19. Sonntag, Y., Zălinescu, C.: Comparison of existence results for efficient points. J. Optim. Theory Appl. **105**(1), 161–188 (2000)
20. Van Nieuwenborgh, D., Vermeir, D.: Preferred answer sets for ordered logic programs. Theory Pract. Log. Program. **6**(1–2), 107–167 (2006)
21. Wagner, D.H.: Semi-compactness with respect to a euclidean cone. Can. J. Math. **29**(1), 29–36 (1977)
22. Zhu, J., Isac, G., Zhao, D.: Pareto optimization in topological vector spaces. J. Math. Anal. Appl. **301**(1), 22–31 (2005)
23. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)